Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 4, pp. 661–669, 1995

0021–8928/95 $24.00+0.00

0021–8928(95)00077-1

ASYMPTOTIC SOLUTION OF THE PROBLEM OF THE INTERACTION OF A PLATE WITH A FOUNDATION INHOMOGENEOUS IN DEPTH†

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(*Received* 26 *May* 1994)

The equations of the contact problem of a circular plate interacting with a half-space inhomogeneous in depth are considered. The Lamé coefficients vary with depth in the half-space according to a general law—that of a gradient (continuously inhomogeneous) or laminated half-space. Use is made of the series expansions of the contact pressures under the plate and of its deflections [1], the terms of the series being the eigenfunctions of the equation of the bending of the plate with boundary conditions representing support conditions for the plate. To solve the dual integral equation, the two-sided asymptotic method of [2, 3] is used. It is proved that, unlike orthogonal polynomial methods and asymptotic methods of the "large λ" and "small λ" types, the method proposed here is effective for both rigid and flexible plates and is asymptotically exact for $\lambda \to 0$ and $\lambda \to \infty$ (λ is a characteristic geometric parameter, defined as the ratio of the thickness of an inhomogeneous layer to the radius of the plate). An analysis is made of how different laws of variation of the Lamé coefficients with depth in the half-space may affect the distribution of contact pressures under the plate, its deflection and the settlement of the surface of the half-space outside the contact zone.

1. The contact problem for the interaction of a circular plate with a half-space inhomogeneous in depth reduces to solving the following system of equations [4]

$$L^0 w(r) = D^{-1}[p(r) - q(r)], \quad 0 \leqslant r \leqslant 1 \tag{1.1}$$

$$\int_0^\infty Q(\alpha) L(\lambda\alpha) J_0(\alpha r) d\alpha = s\lambda w(r), \quad 0 \leqslant r \leqslant 1$$

$$\int_0^\infty Q(\alpha) J_0(\alpha r) d\alpha = 0, \quad r > 1 \tag{1.2}$$

$$r = r'/R, \quad s = \Theta_0 R^3 / D, \quad \Theta_0 = \Theta(0)$$

$$\Theta(z) = 2M(z)(M(z) + \Lambda(z)) / (2M(z) + \Lambda(z))$$

where L^0 is the differential operator of the bending of the plate in a cylindrical system of coordinates, $p(r)$ is the distributed load, $q(r)$ is the contact stresses under the plate and $w(r)$ is its deflections. The plate, of radius R, rests freely on an isotropic half-space, whose Lamé coefficients vary with depth according to the law

$$\Lambda = \Lambda_0(z), \quad M = M_0(z), \quad -H \leqslant z \leqslant 0$$

$$\Lambda = \Lambda_0(-H), \quad M = M_0(-H), \quad -\infty < z < -H$$

where $\Lambda_0(z)$ and $M_0(z)$ are arbitrary continuous functions of depth (the variable z), $\lambda = H/R$ is a characteristic geometrical parameter, s is a parameter characterizing the bending stiffness of the plate and D is the bending stiffness of the plate.

The function $w(r)$ must satisfy the free-edge conditions on the contour of the plate

$$r = 1, \quad \frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} = 0, \quad \frac{d}{dr}\Delta w = 0 \tag{1.3}$$

where ν is Poisson's ratio of the plate and Δ is the Laplacian in polar coordinates; the function must, moreover, be bounded at the origin together with a differential expression corresponding to the bending moment.

In the general case of arbitrary continuous inhomogeneity, the transform of the kernel $L(\alpha)$ is constructed by the numerical method proposed in [5]; on the assumption that

†*Prikl. Mat. Mekh.* Vol. 59, No. 4, pp. 688–697, 1995.

$$\min_{z\in(0;-\infty)} \Theta(z)\geq c_1>0,\quad \max_{z\in(0;-\infty)} \Theta(z)\leq c<\infty,\quad \lim_{z\to-\infty} \Theta(z)=\text{const} \tag{1.4}$$

it possesses the following properties [5, 6]

$$L(\alpha)=A+B|\alpha|+O(\alpha^2),\ \ \alpha\to 0,\ \ A=\lim_{z\to-\infty}\Theta(0)/\Theta(z)$$

$$L(\alpha)=1+C|\alpha|^{-1}+O(\alpha^{-2}),\ \ \alpha\to\infty,\ \ B,C=\text{const} \tag{1.5}$$

2. We shall assume an acquaintance with the definitions of the classes of functions $C^k(0, 1)$, $L^2(0, 1)$ [7], $C_{1/2}^{(0)+}(-1, 1)$ [8]. Let $f\in C^4(0, 1)$ (or, more generally, $f\in L^2(0, 1)$), and suppose that the function satisfies conditions (1.3). Then it can be represented in [0, 1] as a series in the natural modes of oscillation of a circular plate with a free edge [9]

$$f(r)=\sum_{m=0}^{\infty} f_m\varphi_m(r),\ \ 0\leq r\leq 1,\ \ f_m=\int_0^1 f(\rho)\varphi_m(\rho)\rho d\rho \tag{2.1}$$

$$\varphi_0=2^{1/2},\ \ \varphi_m(r)=A_m[J_0(k_m r)-B_m I_0(k_m r)]$$
$$B_m=J_1(k_m)/I_1(k_m) \tag{2.2}$$

The values of A_m and k_m, $m = 0, 1, \ldots, 10$, are given in [1]. If $f\in L^2(0, 1)$, we understand (2.1) to mean that

$$\lim_{k\to\infty}\left\| f-\sum_{m=0}^{k} f_m\varphi_m(r)\right\|_{L^2_{(0,1)}}=0,\ \ \|f\|^2_{L^2_{(0,1)}}=\sum_{m=0}^{\infty} f_m^2$$

which is Parseval's equality.

Let us assume that the deflection function of the plate can be expanded in series (2.1)

$$w(r)=\sum_{m=0}^{\infty} w_m\varphi_m(r),\ \ 0\leq r\leq 1;\ \ w_m=\int_0^1 w(\rho)\varphi_m(\rho)\rho d\rho \tag{2.3}$$

Taking linearity into account, we conclude that the expression for the contact pressures is a linear combination of particular solutions $q_m(r)$ with the same coefficients w_m as for the deflection functions $w(r)$ in (2.3)

$$q(r)=\sum_{m=0}^{\infty} w_m q_m(r),\ \ 0\leq r\leq 1 \tag{2.4}$$

The particular solutions $q_m(r)$ ($m = 0, 1, \ldots$) are determined from the integral equation (1.2). The method of construction and the actual form of the functions $q_m(r)$ may be found in [10]. We shall say that $L(\alpha)$ belongs to class $\Pi_N(\Sigma_M, \Sigma_{N,M})$ if the following relations hold, respectively

$$L(\lambda\alpha)=\begin{cases}\prod_{i=1}^{N}(\alpha^2+A_i^2\lambda^{-2})(\alpha^2+B_i^2\lambda^{-2})^{-1}\equiv L_N(\lambda\alpha)\in\Pi_N\\ \sum_{k=1}^{M}C_k\lambda^{-1}|\alpha|(\alpha^2+D_k^2\lambda^{-2})\equiv L_M^{\Sigma}(\lambda\alpha)\in\Sigma_M\\ L_N(\lambda\alpha)+L_M^{\Sigma}(\lambda\alpha)\in S_{N,M}\end{cases}$$

where A_i, B_i ($i = 1, 2, \ldots, N$), C_k, D_k ($k = 1, 2, \ldots, M$) are certain constants, $(A_i-A_k)(B_i-B_k)\neq 0$, $i\neq k$.

It was proved in [8] that if $L(\alpha)$ possesses properties (1.4) it can be approximated by expressions of the form

$$L(\lambda\alpha)=L_N(\lambda\alpha)+L_\infty^{\Sigma}(\lambda\alpha) \tag{2.5}$$

Henceforth, an integral operator corresponding to $L(\alpha)$ and belonging to class X will also be denoted by X.

Using (2.5), we rewrite (1.2) in operator form as

$$\Pi_N q+\Sigma_\infty q=f \tag{2.6}$$

Corresponding to the operator Π_N in (2.6) we have a function $L(\alpha)$ of class Π_N, and to Σ_N—a function $L(\alpha)$ of class Σ_N, $M=\infty$.

We shall say that Eq. (1.2) satisfies condition A if, whenever $L(\alpha)\in\Pi_N$, one can construct a closed solution following [11]. We shall denote this solution by

$$q^N = \Pi_N^{-1} f \tag{2.7}$$

In other words, condition A means that, for functions $f(x)$ of some class $W(c, d)$, a function $q(x)$ of some class $V(c, d)$ exists such that (2.7) holds. It follows from (2.7) that

$$\|q^N\|_{V(c,d)} \leqslant m(\Pi_N)\|f\|_{W(c,d)}, \quad m(\Pi_N) = \text{const} \tag{2.8}$$

where $m(X)$ denotes some constant which depends on the actual form of the function in X.

It has been proved [8] that, if conditions (1.4) are satisfied, Eq. (1.2) has a unique solution in the space $C_{1/2}^{(0)+}(-1, 1)$ for $\varphi_m(r)$ of type (2.2) for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^a$, where λ^* and λ^a are certain fixed values of λ, and moreover

$$\|q(r)\|_{C_{1/2}^{(0)+}(-1,1)} \leqslant m(\Pi_N, \Sigma_\infty) M_\varphi(-1,1) \tag{2.9}$$

Thus, λ may be chosen in such a way that the operator $\Pi_N^{-1}\Sigma_\infty$ is a contraction [7], and expression (2.7) is an asymptotically exact solution of Eq. (2.6) as $\lambda \to 0$ and $\lambda \to \infty$.

3. Let us replace $w(r)$ on the right of Eq. (1.2) by the mth eigenfunction $\varphi_m(r)$. Using previous results [8], one can write down a closed approximate solution of Eq. (1.2) in the form (2.7), relative to $q_m^N(r)$ [10].

The contact pressures $q_m^N(r)$, in turn, may be written as series (2.1). We have

$$q_m^N(r) = \sum_{j=0}^{\infty} \Sigma_j^m \varphi_j(r), \quad \Sigma_j^m = \int_0^1 q_m^N(r)\varphi_j(\rho)\rho d\rho \tag{3.1}$$

The actual form of Σ_j^m is shown in [10].

We shall assume that $p(r) \in C^4(0, 1)$ (or, in a more general form, $p(r) \in L^2(0, 1)$), i.e. it can be expressed as a series

$$p(r) = \sum_{m=0}^{\infty} p_m \varphi_m(r), \quad p_m = \int_0^1 p(\rho)\varphi_m(\rho)\rho d\rho \tag{3.2}$$

Then, substituting expansions (2.3), (3.1) and (3.2) into (1.1), we obtain an infinite system of linear algebraic equations for the coefficients w_m, which can be written in canonical form as [7]

$$w_m - \frac{a}{k_m^4}\sum_{j=0}^{\infty} w_j \mathrm{E}_j^m = \frac{p_m}{k_m^4}, \quad m = 0, 1, 2, \ldots; \; a = -1 \tag{3.3}$$

We have

$$\sum_{m=0}^{\infty}\left|\frac{p_m}{k_m^4}\right|^2 \leqslant M\sum_{m=0}^{\infty}|p_m|^2 < \infty$$

given the restrictions imposed on $p(r)$.

It follows from estimate (2.9) and the equilibrium conditions of the plate that

$$\int_0^{2\pi}\int_0^1 q(r) r dr d\varphi = \int_0^{2\pi}\int_0^1 p(r) r dr d\varphi$$

We have

$$\sum_{m=0}^{\infty} k_m^{-8}\sum_{j=0}^{\infty}\left|\mathrm{E}_j^m\right|^2 \leqslant \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\left|\mathrm{E}_j^m\right|^2 \leqslant M(\Pi_N, \Sigma_\infty)\sum_{m=0}^{\infty}|p_m|^2 < \infty$$

Thus, by Theorem 3a of [7, p. 503], if $a = -1$ is not an eigenvalue of system (3.3), the system may be solved by the reduction method (replacing it by a system of n equations in n unknowns)

$$w_m - \frac{a}{k_m^4}\sum_{j=0}^{n-1} w_j \mathrm{E}_j^m = \frac{p_m}{k_m^4}, \quad m = 0, 1, \ldots, n-1; \; a = -1 \tag{3.4}$$

where, for sufficiently large n, system (3.4) is solvable and the approximate solutions converge to the exact solution.

We have thus proved the following theorem.

Theorem 1. Let conditions (1.4) hold and let $p(r) \in V_i(0, 1)$, $i = 0$ or $i = 1$ ($V_1(0, 1) \equiv C^4(0, 1)$, $V_2(0, 1 \equiv L^2(0, 1))$. Then the system of Eqs (1.1) and (1.2) is uniquely solvable for $w(r) \in V_i(0, 1)$ ($i = 0$ or $i = 1$), $q(r) \in C_{1/2}^{(0)+}(-1, 1)$ for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^a$, where λ^* and λ^a are certain fixed values of λ. Under these conditions the following estimates hold

$$\|w(r)\|_{V_i(0,1)} \leq m(\Pi_N, \Sigma_\infty)\|p(r)\|_{V_i(0,1)}$$

$$\|q(r)\|_{C_{1/2}^{(0)+}(-1,1)} \leq m_1(\Pi_N, \Sigma_\infty)\|p(r)\|_{V_i(0,1)}$$

4. We will now proceed to an asymptotic determination of the settlement in the surface of the half-space outside the contact zone. Equation (1.2) was obtained by using the following representation for the vertical displacements of the surface of the half-space

$$f(r) = \frac{1}{s\lambda}\int_0^\infty Q(\alpha)L(\lambda\alpha)J_0(\alpha r)d\alpha \tag{4.1}$$

Let us find an analytic expression for this function $f(r)$ when $L(\alpha) \in \Pi_N$, $r > 1$. If $L(\alpha) \in \Pi_N$, the function $Q^N(\alpha)$ is obtained in analytic form when constructing a solution of Eq. (1.2). Using formula (4.1), we obtain an expression for the functions $f(r)$, $r > 1$, if $L(\alpha) \in \Pi_N$, which we denote by $f^N(r)$. By (2.4), $Q^N(\alpha)$ may be written as

$$Q^N(\alpha) = \sum_{m=0}^{\infty} w_m Q_m^N(\alpha)$$

The corresponding expressions for the functions $f_m^N(r)$ are

$$f_m^N(r) = \frac{2}{\pi}A_m\{G(r,k_m) - B_m G(r,ik_m) +$$

$$+\sum_{n=1}^{N} D_n b_n \lambda^{-1} I_n\left[\Psi_n(k_m) - B_m\Psi_n(ik_m) + \sum_{j=1}^{N} C_j^m \gamma_n(ia_j\lambda^{-1})\right]\Bigg\}, \quad r>1, \quad m = 0,1,2,\ldots \tag{4.2}$$

$$\Psi_n(a) = L_N^{-1}(\lambda a)\gamma_n(a), \quad \gamma_n(a) = \frac{a\sin a}{b_n^2\lambda^{-2} + a^2}$$

$$I_n = \int_1^r \frac{\exp[\lambda^{-1}b_n(t-1)]}{\sqrt{r^2 - t^2}}dt, \quad G(Ar) = \int_0^1 \frac{\cos At}{\sqrt{r^2 - t^2}}dt$$

$$D_n = (a_n^2 b_n^{-2} - 1)\prod_{\substack{i=1 \\ i\neq n}}^{N} \frac{-b_n^2 + a_i^2}{-b_n^2 + b_i^2}$$

Finally, we obtain

$$f^N(r) = \sum_{m=0}^{M} w_m f_m^N(r), \quad r > 1 \tag{4.3}$$

where w_m are the same coefficients as in (2.3).

The question arises of the use of formulae (4.2) and (4.3) to determine the settlement of the surface of an inhomogeneous half-space, i.e. when the transform of the kernel $L(\alpha)$ possesses properties (1.5) and $L(\alpha)$ belong to class $S_{N,M}$. By Theorem 1, solution (2.7), represented by formulae (3.1) and (2.4), is an asymptotically exact solution of Eq. (1.2) when the operators $L(\alpha)$ belong to class $S_{N,M}$ as $\lambda \to 0$ and $\lambda \to \infty$. Substituting this asymptotic solution q^N into (1.2) for $L(\alpha) \in S_{N,M}$, we find an approximate solution for the settlement of the surface of the inhomogeneous half-space outside the contact zone, $f^s(r)$ in the general case, when $L(\alpha) \in S_{N,M}$

$$f^s(r) = (\Pi_N + \Sigma_M)q^N(r), \; r > 1$$

The asymptotic properties of the solution of system (1.1), (1.2) are established in the finite interval $r \in [0, 1]$. We shall show that these asymptotic properties are preserved when changing to the approximately determined function $f(r)$, $r \in (1, \infty)$.

We may assume without loss of generality that $M = 1$. In that case

$$\Sigma_1 q^N(r) = \int_0^1 q^N(\rho)\rho\left[\int_0^\infty \frac{C\lambda^{-1}\alpha}{\alpha^2 + D^2\lambda^{-2}} J_0(\alpha s)J_0(\alpha\rho)d\alpha\right]d\rho =$$

$$= C\lambda^{-1}K_0(rD\lambda^{-1})\int_0^1 q^N(\rho)I_0(\rho D\lambda^{-1})\rho d\rho, \quad r > 1$$

Using the asymptotic properties of cylindrical functions of an imaginary argument, we obtain the estimates

$$\max_{r>1}\left|\Sigma_1 q^N(r)\right| \leq M^* \exp(-D\lambda^{-1}\delta), \quad \lambda \to 0, \quad (\lambda > \lambda^*), \quad \delta = r - \rho > 0$$

$$\max_{r>1}\left|\Sigma_1 q^N(r)\right| \leq M^0\lambda^{-1+\varepsilon}, \quad \lambda \to \infty \ (\lambda > \lambda^0), \quad \varepsilon > 0, \quad \varepsilon \to 0$$

where the constants M^* and M^0 are independent of λ. This implies the following theorem.

Theorem 2. Formulae (4.2) and (4.3) provide an asymptotically exact representation of the settlement of the surface of a half-space inhomogeneous in depth outside the contact zone under the assumptions of Theorem 1, for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^0$, where λ^* and λ^0 are certain fixed values of λ.

Remark. Analogous results hold for the bending of a beam resting on a strip inhomogeneous in depth or on an inhomogeneous half-space. The proof uses the asymptotic properties of the approximate solutions of the appropriate contact problems, as established in [2, 3, 12].

6. As an example, consider the bending of a circular plate under a uniformly distributed load of unit intensity. In that case ($p(r) = p =$ const)

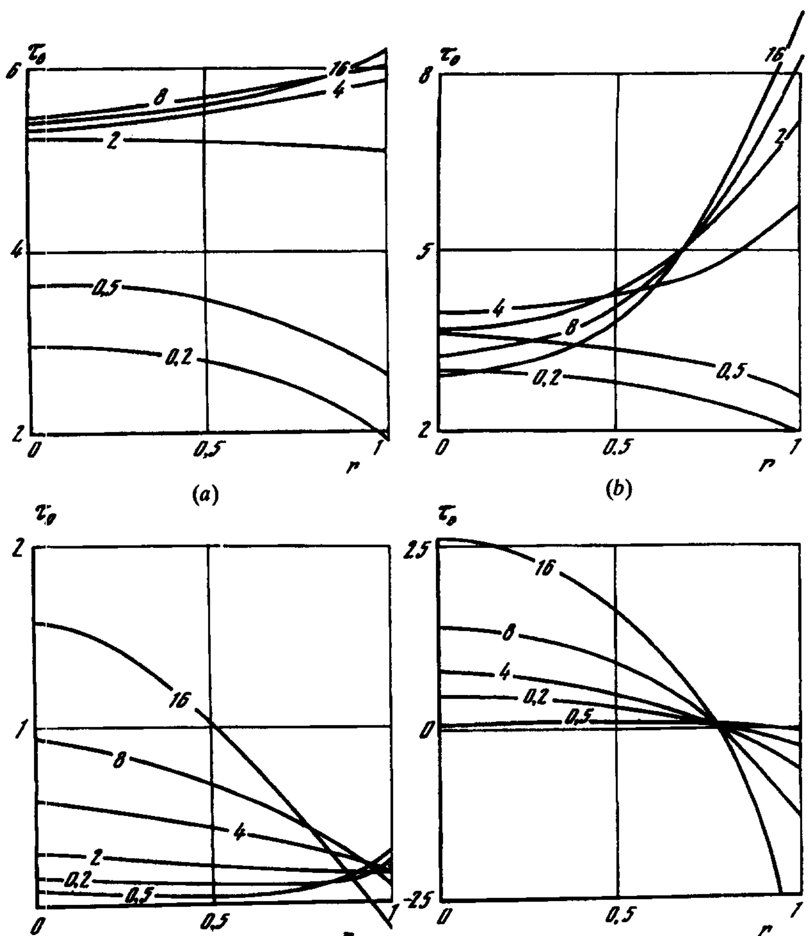

Fig. 1(a–d).

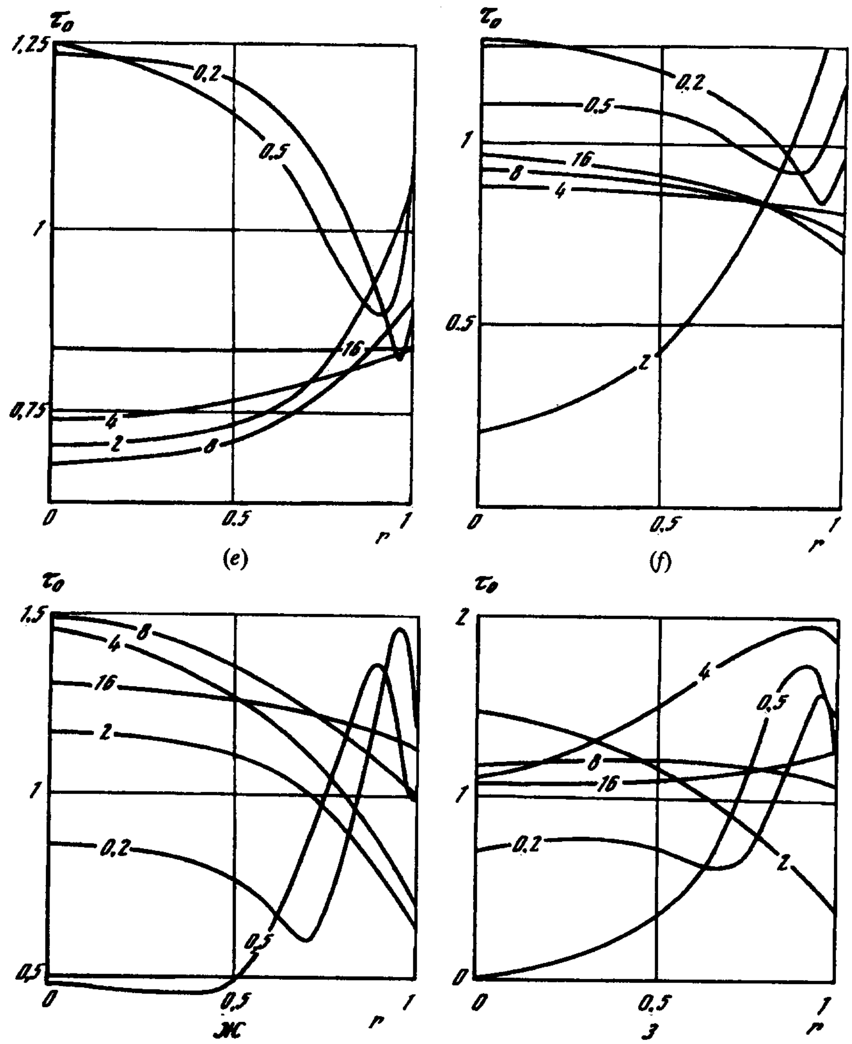

Fig. 1(e–h).

$$p_0 = p\sqrt{2}/2; \quad p_m = 0, \quad m = 1, 2, \ldots$$

The plate rests on a half-space whose Young's modulus varies with depth according to the law

$$E(z) = E_0\varphi(z), \quad -1 \leqslant z \leqslant 0; \quad E(z) = E_0\varphi(-1), \quad z < -1$$

Poisson's ratio of the foundation is $\nu_0 = 1/3$, $-\infty < z \leqslant 0$ and that of the plate $\nu_n = 0.15$.
We consider the following types of inhomogeneity.
1. Monotonic (power-law)
(a) increasing with depth

$$\varphi_1(z) = 0{,}1 - z^{2a_k}, \quad a_k = \ln 0{,}1(k-1)/2\ln 0{,}5, \quad k = 2$$

(b) decreasing

$$\varphi_2(z) = 1{,}1 - z^{2a_k}, \quad a_k = \ln(1{,}1 - 0{,}1k)/2\ln 0{,}5, \quad k = 9$$

2. Non-monotonic (sinusoidal)

$$\varphi_3(z) = 1{,}1 + \sin(\pi z), \quad \varphi_4(z) = 0{,}1 - \sin(\pi z)$$

Figure 1 shows graphs of the quantity $\tau^0(r) = q_n(r)q_0^{-1}(r)$, which characterizes the distribution of the contact normal pressures under the plate on the inhomogeneous foundation compared with a homogeneous base $q_0(r)$, for different values of λ ($q_0(r)$ corresponds to $E(z) \equiv E_0$, $z < 0$). Here and below Fig. 1(a), (c), (e) and (g) correspond to a plate bending stiffness $s = 0, 1$; Fig. 1(b), (d), (f) and (h) correspond to $s = 3$. The digits on the curves indicate the values of λ. The ordinate τ^0 in Fig. 1(a) and (b) corresponds to the law $\varphi_1(z)$, in Fig. 1(c) and (d) to $\varphi_2(z)$, in Fig. 1(e) and (f) to $\varphi_3(z)$ and in Fig. 1(g) and (h) to $\varphi_4(z)$. We can conclude that in the case of an inhomogeneity which decreases monotonically with depth, as in $\varphi_2(z)$, negative contact pressures occur, indicating that the plate separates from the foundation (Fig. 1g and h). In such cases the formulation of the problem must be modified. The plate/foundation contact zone may be determined from the condition that the contact pressures vanish at the boundaries of the zone. The separation zone expands when the bending stiffness of the plate is increased (Fig. 1h).

For the non-monotonic inhomogeneity laws $\varphi_3(z)$ and $\varphi_4(z)$ it is characteristic that when $\varphi(z)$ at the surface of the foundation increases with depth ($\varphi_3(z)$), one observes an increase in the quantity τ^0 characterizing the coefficient of the contact stress singularity as the plate edge is approached from within. The plate does not separate from the base.

If $\varphi(z)$ at the surface of the foundation decreases with depth ($\varphi_4(z)$), there is a decrease in τ^0 as the plate edge is approached from within. It is evident that the distribution of the contact pressures depends essentially both on the depth of the inhomogeneous layer and the form of inhomogeneity, and on the bending stiffness of the plate.

Figure 2 shows graphs representing the relative settlement of the surface of the inhomogeneous half-space relative to that of a homogeneous base (under the plate and outside the contact zone)

$$\Delta(r) = w_n(r) w_0^{-1}(r), \quad 0 \leq r \leq 1; \quad \Delta(r) = f_n(r) f_0^{-1}(r), \quad r > 1$$

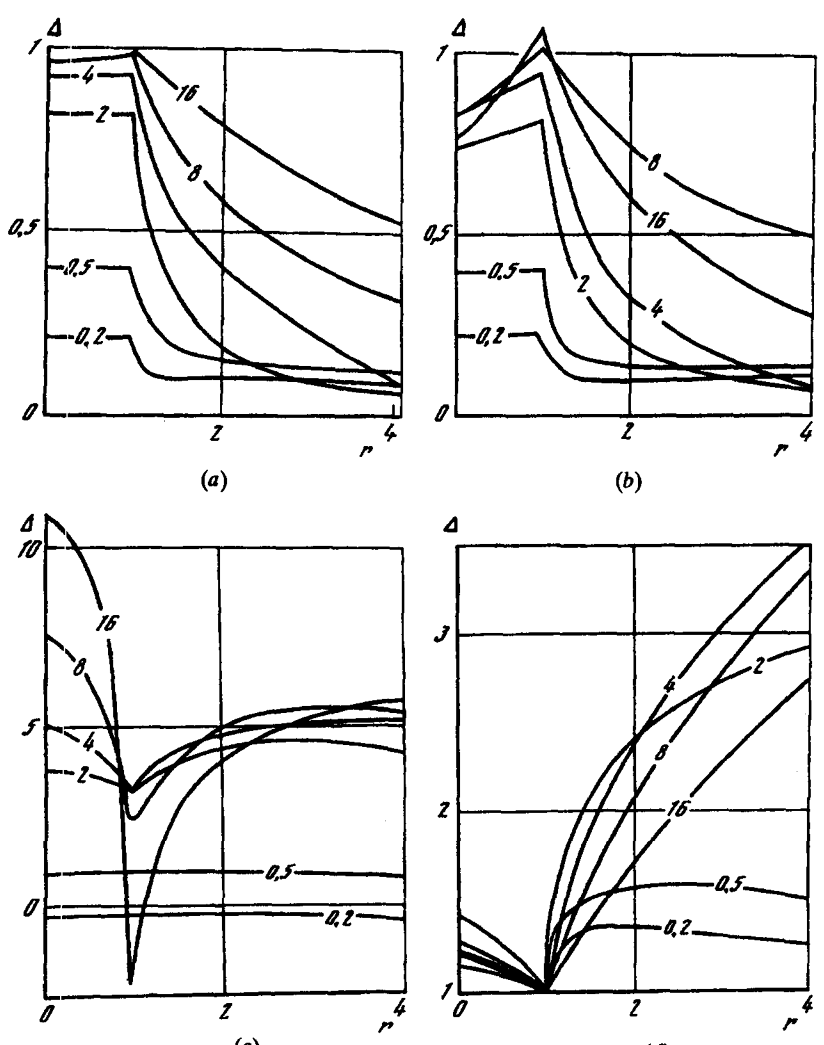

Fig. 2(a–d).

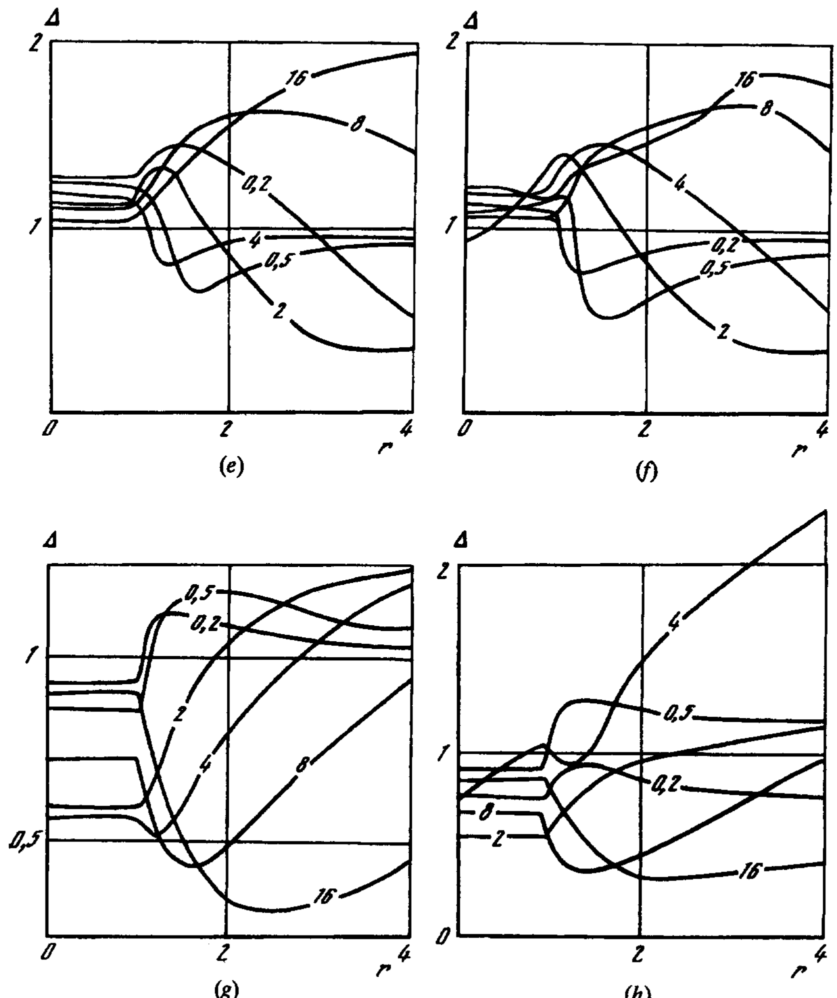

Fig. 2(e–h).

When $\varphi(z)$ is a monotonically increasing function (inhomogeneity law $\varphi_1(z)$), the "cratering" of the settlement of the base surface under the plate is steeper than for a homogeneous base (Fig. 2a, b). Conversely, when $\varphi(z)$ is monotonically decreasing (inhomogeneity law $\varphi_2(z)$), the settlement cratering is shallower than for a homogeneous foundation (Fig. 2c, d). For the non-monotonic inhomogeneity laws $\varphi_3(z)$ and $\varphi_4(z)$, the shape of the cratering depends essentially on λ—the relative thickness of the inhomogeneous layer under the plate (Fig. 2e–h).

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Translated by D.L.